Int. J. Heat Mass Transfer. Vol. 8, pp. 915–920. Pergamon Press 1965. Printed in Great Britain

TEMPERATURE DISTRIBUTIONS IN RADIATING HEAT SHIELDS BY THE METHOD OF SINGULAR PERTURBATIONS

H. F. MUELLER and N. D. MALMUTH

North American Aviation, Los Angeles Division, International Airport, Los Angeles, Calif.

(*Received* 21 *September* 1964 *and in revised form* 5 *January* 1965)

Abstract—The temperature distribution on a thin heat shield shell subject to longitudinal conduction, radiative losses, and an arbitrary aerodynamic source loading is considered. For the case where the internal face and the ends are insulated, the determination of the normalized temperatures reduces to a two-point boundary value problem for a nonlinear second-order equation in which the radiation–conduction parameter, ϵ, appears explicitly. Uniformly valid solutions are obtained for small values of ϵ by singular perturbation methods. Because of the presence of conduction, these distributions show the expected uniformization tendency relative to the $\epsilon = 0$ solution, as well as the anticipated "boundary layer" structure near the insulated ends. Factors influencing the range of applicability of the solution as well as extensions of the analysis are discussed.

NOMENCLATURE

k, thermal conductivity [kcal/m h];

q, aerodynamic heat transfer distribution function normalized with respect to $\sigma \bar{\epsilon} \bar{T}^4 = \bar{q}$;

$\bar{q}$, maximum value of aerodynamic heat flux along shield [kcal/m² h];

x, coordinate along shield normalized with respect to L;

L, length of shield [m] (see Fig. 1);

T, temperature normalized with respect to $\bar{T}$;

$\bar{T}$, maximum value of temperature along shield (grade);

$\epsilon = \dfrac{\delta k}{\sigma \bar{\epsilon} L^2 \bar{T}^3}$ conduction radiation ratio;

$\bar{\epsilon}$, surface emissivity;

σ, Stefan–Boltzmann constant [kcal/m² h (deg)⁴].

Subscripts

outer, valid in outer region;

inner, valid in inner region, i.e. near ends, (0, 1).

Superscripts

(L), left-hand inner region;

(R), right-hand inner region.

INTRODUCTION

Heat shield shells have been used extensively as a means of thermal protection of spacecraft during re-entry. A typical application is as indicated in Fig. 1, where a cross section of a wing is shown shielded by the shell AOC, of constant thickness δ. The thermal stresses created are related to the degree of chordwise variation of the temperature along the shell. By use of a highly conducting material for the shell, it is possible to reduce this variation, and hence effect a saving in required structural weight. In a more general concept, it has been proposed [1] to vary sectional properties, such as the material composition and thickness, as a function of x, the coordinate along the shield length, to obtain further benefits in structural efficiency. Therefore, actual quantitative design assessments of the structural weight reduction afforded by the use of the conduction effects described previously, depend on accurate and quick methods for the estimation of the temperature distributions.

In the present paper, singular perturbation techniques are applied to obtain closed form solutions of the temperature distribution for the case of small conductive transfer in comparison to radiation losses and convective loading. For the case at hand, the ends AB and CD, as well as the internal face BPD, are considered

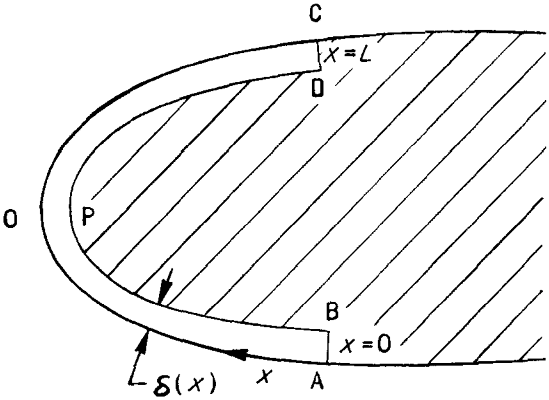

FIG. 1. Heat shield geometry.

insulated. In addition, in accord with general practice, the thickness ratio, δ/L, is regarded as small, and as a result, variations of the temperature across the thickness may be neglected. Furthermore, it is assumed for convenience, that the section properties are constant with x. The extension of the analysis corresponding to the last assumption to the general case of section properties which are arbitrary functions of x is straightforward. Because of the high temperatures anticipated, it is assumed that the surroundings are at a negligible temperature in comparison to those of the shield. In addition, each surface element of the shield is considered to be radiating away heat energy according to Stefan's law, and, to have achieved steady-state conditions.

Based on the foregoing assumptions, it can be easily shown that the boundary value problem for the temperature corresponding to the assumed one-dimensional transfer model is:

$$\epsilon T'' - T^4 + q(x) = 0 \qquad (1a)$$

$$T'(0) = T'(1) = 0 \qquad (1b)$$

where the primes denote differentiation with respect to x.

Problems of this type have arisen in other physical situations. Carslaw [2] describes the case of a thermally radiating wire carrying an electric current. In his discussion, $q(x)$ is a constant. Lick [3] treats the case for which q is linear in x, in connection with a radiating gas between parallel plates, and with the temperature, rather than the heat flux, specified at the boundaries. Hrycak [4] considers another special forcing function representing solar radiation. As mentioned previously, the case of weak conduction will be treated here, i.e. $\epsilon \ll 1$. This situation is typical of a thin shield exposed to a re-entry environment. As a result, it is anticipated that in an "outer" region, i.e. away from the ends $x = 0, 1$, the temperature, T, has the following asymptotic representation:

$$T(x;\, \epsilon) = T_{\text{outer}} = [q(x)]^{1/4} + \epsilon T_1(x) + \ldots, \quad \text{as } \epsilon \downarrow 0 \qquad (2)$$

The expansion, (2), is nonuniformly valid with respect to x by virtue of the fact that the conduction must balance the other modes of transfer in an "inner" region or "boundary layer" near the ends, in order for the temperature distribution to satisfy the insulation boundary conditions, (1b).

Because of this situation, the boundary value problem given by equation (1) is a singular perturbation problem. Although Lick [3] recognized this fact independently of the present authors in connection with the somewhat different but related problem, described earlier, he obtained only the first term of the "inner" expansion of the solution and not the uniformly valid solution. The latter will be obtained in the present paper using matching principles embodied in the singular perturbation method and described in reference 5. Moreover, by contrast to references 3 and 4, the present analysis will treat arbitrary q functions, subject only to the restriction that neither $q(0)$ nor $q(1) = 0$. The latter case requires special treatment because the form of the expansions changes. Before proceeding, it is noted that if either or both $q'(0)$ and $q'(1) = 0$, then no inner expansion is required and the solution is the regular perturbation development, (2). This is the case treated in reference 4.

TEMPERATURE SOLUTION

Substitution of the outer expansion, (2) into (1) and equating terms of like orders gives:

$$T_1 = \frac{1}{4} q^{-3/4} [q^{1/4}]'' \qquad (3)$$

In the inner region, on the basis of previous considerations, it is asserted that the term $\epsilon T''$ is of the same order as the other terms in equation (1a). This implies that the "left-hand" inner region is given by $x = O(\sqrt{\epsilon})$. The foregoing suggests that the left-hand inner representation is given by:

$$T = T^{(L)}_{\text{inner}}(x;\epsilon) = g_0(\tilde{x}) + \sqrt{(\epsilon)}\, g_1(\tilde{x}) + \epsilon g_2(\tilde{x}) + \ldots \quad (4)$$

as $\epsilon \downarrow 0$

where:

$$\tilde{x} = \frac{x}{\sqrt{\epsilon}} \quad (5)$$

In this region:

$$q(x) = q[(\sqrt{\epsilon})\,\tilde{x}] = q(0) + (\sqrt{\epsilon})\,\tilde{x}q'(0) + \frac{\epsilon\tilde{x}^2}{2} q''(0) + \ldots \quad (6)$$

Substitution of equations (4) and (6) into equation (1), and retaining terms of $O(1)$ gives the following approximate equation and boundary conditions for g_0, where the primes denote differentiation with respect to $\tilde{x}$:

$$g_0'' - g_0^4 + q(0) = 0 \quad (7a)$$

$$g_0'(0) = g_0'(1) = 0 \quad (7b)$$

Quadratures may be applied to obtain the following general solution of equation (7):

$$\tilde{x} = \frac{1}{\sqrt{2}} \int_{g_0(0)}^{g_0} \left\{ \frac{\xi^5 - g_0^5(0)}{5} - q(0)\,[\xi - g_0(0)] \right\}^{-1/2} d\xi \quad (8)$$

The alternative singular solution [6] is the degenerate one, i.e.:

$$g_0^4 = q(0) \quad (9)$$

which is the envelope formed by variation of the parameter $g_0(0)$ in (8). The solution, (8), implies that g_0 approaches a point of infinite discontinuity for $\tilde{x} \to a$, where a is a finite number. Since such a discontinuity is physically unreasonable, (8) is discarded in favor of (9). By use of the latter solution, and application of the previously outlined procedure, the following problems for the higher order approximate quantities are formulated:

$$O(\sqrt{\epsilon}):\; g_1'' - \omega^2 g_1 + q'(0)\,\tilde{x} = 0 \quad (10a)$$

$$g_1'(0) = 0 \quad (10b)$$

$$\omega^2 \equiv 4[q(0)]^{3/4} \quad (10c)$$

$$O(\epsilon):\; g_2'' - 4\,g_0\,g_2 - 6\,g_0^2\,g_1^2 + \frac{\tilde{x}^2}{2} q''(0) = 0 \quad (11a)$$

$$g_2'(0) = 0 \quad (11b)$$

so that:

$$T^{(L)}_{\text{inner}} = [q(0)]^{1/4} + \sqrt{\epsilon}\left[g_1(0)\cosh\omega\tilde{x} + \frac{\tilde{x}}{\omega^2} q'(0) - \frac{q'(0)}{12\omega^3}\sinh\omega\tilde{x}\right] + \epsilon\left\{A\left[\frac{\exp[-2\omega\tilde{x}]}{3\omega^2} - 1/2\left(1/2\omega^2 + \frac{\tilde{x}}{\omega} + \tilde{x}^2\right)\exp[-\omega\tilde{x}] + \frac{11}{12\omega^2}\sinh\omega\tilde{x}\right] - \frac{C}{\omega^2}(2/\omega^2 + \tilde{x}^2) + D\cosh\omega\tilde{x}\right\} + O(\epsilon\sqrt{\epsilon}) \quad (12)$$

By similar considerations, applied to the layer, $1 - x = O(\sqrt{\epsilon})$, the "right-hand" boundary layer solution is found to be:

$$T^{(R)}_{\text{inner}} = [q(1)]^{1/2} + \sqrt{\epsilon}\left[h_1(0)\cosh\tau x^\dagger - \frac{x^\dagger}{\tau^2} q'(1) + \frac{q'(1)}{\tau^3}\sinh\tau x^\dagger\right] + \epsilon\left\{F\left[\frac{\exp[-2\tau x^\dagger]}{3\tau^2} - 1/2\left(\frac{1}{2\tau^2} + \frac{x^\dagger}{\tau} + x^{\dagger 2}\right)\exp[-\tau x^\dagger] + \frac{11}{12\tau^2}\sinh\tau x^\dagger\right] - \frac{G}{\tau^2}(2/\tau^2 + x^{\dagger 2}) + H\cosh\tau x^\dagger\right\} + O(\epsilon\sqrt{\epsilon}) \quad (13)$$

where:

$$h_0 = [q(1)]^{1/4}, \quad \tau^2 \equiv 4h_0^3, \quad x^\dagger \equiv \frac{1-x}{\sqrt{\epsilon}}$$

$$A \equiv 6g_0^2\,[q'(0)]^2/\omega^6, \quad C \equiv A\omega^2 - q''(0)/2$$

$$F = 6h_0^2\,[q'(1)]^2/\tau^6, \quad G \equiv F\tau^2 - q''(1)/2$$

The constants, $g_1(0)$, $h_1(0)$, D, H are determined by matching the inner solutions, $T^{(L)}_{\text{inner}}$, $T^{(R)}_{\text{inner}}$, with the outer one, T_{outer}. General principles of matching are discussed in reference 5. In the present case, it is sufficient to require that the outer expansion, (2), written in terms of the approximate inner variables, agrees with both inner expansions for large values of the respective inner variables to suitable orders of ϵ. This gives:

$$g_1(0) = q'(0)/\omega^3 \qquad (14a)$$

$$D = -11A/12\omega^2 \qquad (14b)$$

$$h_1(0) = -q'(1)/\tau^3 \qquad (14c)$$

$$H = -11F/12\tau^2 \qquad (14d)$$

Having obtained the outer and the two inner expansions, one can readily derive, using the method described in reference 5, the composite uniformly valid expansion. Thus, equations (12), (13), and (14) give for this expansion:

$$T(x;\epsilon) = [q(x)]^{1/4} + (\sqrt{\epsilon})\left[\frac{q'(0)}{\omega^3}\exp[-\omega\tilde{x}] - \frac{q'(1)}{\tau^3}\exp[-\tau x^\dagger]\right]$$
$$+ \epsilon\left\{A\left[\frac{\exp[-2\omega\tilde{x}]}{3\omega^2} - 1/2\left(\frac{7}{3}\omega^2 + \tilde{x}/\omega + \tilde{x}^2\right)\exp[-\omega\tilde{x}]\right]\right.$$
$$+ F\left[\frac{\exp[-2\tau x^\dagger]}{3\tau^2} - 1/2\left(\frac{7}{3}\tau^2 + x^\dagger/\tau + x^{\dagger 2}\right)\exp[-\tau x^\dagger]\right]$$
$$\left. + \frac{[q^{1/4}(x)]''\,q^{-3/4}(x)}{4}\right\} + O(\epsilon\sqrt{\epsilon}) \qquad (15)$$

To show the qualitative trends of the above solution, calculations were made of the temperature distributions over circular shells, using a q function determined empirically by Tewfik and Giedt [7] from experiments on cylinders, i.e.:

$$q = 0{\cdot}37 + 0{\cdot}48\sin\pi x - 0{\cdot}15\cos 2\pi x \qquad (16)$$

Although equation (16) was derived for the heat-transfer coefficient, it has been used above to a good approximation for the heat flux variation, since at hypersonic speeds, the wall temperatures are generally small compared with the recovery temperature. The latter quantity is approximately constant between $x = 0$ and $x = 1$.

The results are plotted in Fig. 2, where the

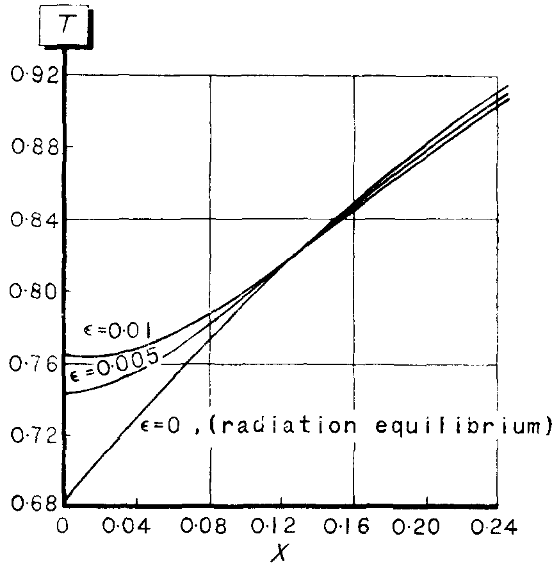

FIG. 2. Temperature solution.

boundary-layer phenomena and the uniformizing effect of conduction relative to the $\epsilon = 0$ solution are evident. Also, it is noted that the upper bound on ϵ from a computational accuracy standpoint in this example appears to be approximately 0·01. This upper bound obviously would change with the inclusion of a greater number of terms in the expansion, and the selection of other q distributions which have different values of $q(0)$ and $q(1)$; the latter quantities appear in inverse powers in the terms of the expansion.

It should be noted that the foregoing developments represent only the first part of a more comprehensive research program. In this respect, attention is being directed to the treatment of the singular case for which $q(0)$ or $q(1) = 0$ and also toward the modification of the previous expansions in order to offset the limiting effects of the aforementioned factors. Further extensions envisioned include treatment of variable section properties, internal radiation, and the removal of the one-dimensional restriction in the analysis. Finally, a numerical solution of equation (1), on a firm foundation with respect to stability and convergence would be desirable to assess the accuracy of (15).

ACKNOWLEDGEMENT

The authors wish to express their appreciation for the helpful suggestions of Dr. J. Kevorkian, and Professors J. D. Cole and G. B. Whitham of the California Institute of Technology.

REFERENCES

1. H. F. Mueller, Hypersonic leading edge design, NAA Report, *NA*-58-277-7 (1958).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed., p. 154. Oxford University Press, London (1960).
3. W. Lick, Energy transfer by radiation and conduction, *Proc. Heat Trans. Fluid Mech. Inst.* pp. 14–24 (1963).
4. P. Hrycak, Influence of conduction on spacecraft skin temperatures, *AIAA J.* **1**, 2619–2621 (1963).
5. S. Kaplun and P. A. Lagerstrom, Asymptotic expansions of the Navier–Stokes solutions for small Reynolds numbers, *J. Math. Mech.* **6**, 585–593 (1957).
6. E. L. Ince, *Ordinary Differential Equations*, pp. 11, 12, 83–88. Dover, New York (1956).
7. O. Tewfik and W. Giedt, Heat transfer, recovery factor, and pressure distributions around a circular cylinder normal to a supersonic rarified-air stream, *J. Aeronaut. Sci.* **27**, 721–728 (1960).

Résumé—La distribution de température dans une coque mince faisant bouclier de chaleur sujette à la conduction longitudinale, aux pertes par rayonnement et à une charge arbitraire d'origine aérodynamique est considérée. Pour le cas où la face intérieure et les extrémités sont isolées, la détermination des températures normalisées se réduit à un problème de conditions aux limites en deux points pour une équation non-linéaire du second ordre dans lequel le paramètre rayonnement-conduction, ϵ, apparaît explicitement. Des solutions valables uniformément sont obtenues pour des petites valeurs de ϵ par des méthodes de perturbation singulières. A cause de la présence des conditions, ces distributions montrent la tendance attendue d'uniformisation relative à la solution $\epsilon = 0$, aussi bien que la structure attendue du type "couche limite" près des extrémités isolées. Les facteurs influençant la gamme d'applicabilité de la solution aussi bien que les extensions de l'analyse sont discutés.

Zusammenfassung—In einem dünnen Wärmeschild, in dem Wärmeleitung in Längsrichtung, Abstrahlverluste und beliebig verteilte Wärmequellen wegen der aerodynamischen Heizung auftreten, wird die Temperaturverteilung untersucht. Für den Fall, dass die Innenseite und die Enden isoliert sind, vereinfacht sich die Bestimmung der dimensionslosen Temperaturen auf ein Zwei-Punkte-Grenzwertproblem für eine nichtlineare Gleichung zweiter Ordnung, in welcher der Strahlungs-Wärmeleitungsparameter ϵ explizit erscheint. Allgemein gültige Lösungen erhält man für kleine Werte von ϵ durch eine singuläre Störungstheorie. Wegen der vorgegebenen Bedingungen zeigen diese Verteilungen die erwartete Vereinheitlichungstendenz relativ zur Lösung $\epsilon = 0$ ebenso wie die erwartete "Grenzschicht"-Struktur in Nähe der isolierten Enden. Es werden Faktoren diskutiert, die den Anwendungsbreich der Lösung und den Erfassungsbereich der Analyse beeinflussen

Аннотация—Рассмотрено распределение температуры на тонкой оболочке теплового экрана при продольной теплопроводности, лучистых потерях и произвольной интенсивности аэродинамических источников. Для случая изолированной внутренней поверхности и концов определение нормированных значений температур приводит к задаче двухточечной краевой задаче для нелинейного уравнения второго порядка, в котором в явном виде имеется излучательно-кондуктивный параметр ϵ. Длямалых значений ϵ методом сингулярных возмущений получены равномерно сходящиеся решения. Из-за наличия условий эти распределения проявляют ожидаемую униформизацию по отношению к решению при $\epsilon = 0$, а также ожидаемую структуру «пограничного слоя» у изолированных концов. Обсуждаются факторы, влияющие на диапазон применимости решения, а также на распространение анализа на другие случаи.